

ANALYTIC MODULI SPACES OF SIMPLE (CO)FRAMED SHEAVES

HUBERT FLENNER AND MARTIN LÜBKE

ABSTRACT. Let X be a complex space and \mathcal{F} a coherent \mathcal{O}_X -module. A \mathcal{F} -(co)framed sheaf on X is a pair (\mathcal{E}, φ) with a coherent \mathcal{O}_X -module \mathcal{E} and a morphism of coherent sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{E}$ (resp. $\varphi : \mathcal{E} \rightarrow \mathcal{F}$). Two such pairs (\mathcal{E}, φ) and (\mathcal{E}', φ') are said to be *isomorphic* if there exists an isomorphism of sheaves $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$ with $\alpha \circ \varphi = \varphi'$ (resp. $\varphi' \circ \alpha = \varphi$). A pair (\mathcal{E}, φ) is called *simple* if its only automorphism is the identity on \mathcal{E} . In this note we prove a representability theorem in a relative framework, which implies in particular that there is a moduli space of simple \mathcal{F} -(co)framed sheaves on a given compact complex space X .

1. INTRODUCTION

Let X be a complex space and \mathcal{F} a coherent \mathcal{O}_X -module. By a \mathcal{F} -coframed sheaf on X we mean a pair (\mathcal{E}, φ) with

- (a) \mathcal{E} is a coherent \mathcal{O}_X -module,
- (b) $\varphi : \mathcal{F} \rightarrow \mathcal{E}$ is a morphism of coherent sheaves.

(Following [HL1], [HL2], a \mathcal{F} -framed sheaf is dually a pair (\mathcal{E}, φ) with \mathcal{E} as above and a morphism $\varphi : \mathcal{E} \rightarrow \mathcal{F}$.) Two such pairs (\mathcal{E}, φ) and (\mathcal{E}', φ') are said to be *isomorphic* if there exists an isomorphism of sheaves $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$ with $\alpha \circ \varphi = \varphi'$. A pair (\mathcal{E}, φ) is called *simple* if its only automorphism is the identity on \mathcal{E} . The purpose of this note is to show that there is a moduli space of simple \mathcal{F} -(co)framed sheaves on a given compact complex space X .

More generally, we will show the following relative result. Let $X \rightarrow S$ be a proper morphism of complex spaces. By a family of \mathcal{F} -coframed sheaves over S (or a \mathcal{F} -coframed sheaf on X/S in brief) we mean a \mathcal{F} -coframed sheaf (\mathcal{E}, φ) on X that is S -flat. Such a family will be called *simple* if its restriction to each fibre $X(s) := f^{-1}(s)$ is simple.

We consider the set-valued functor $P : \mathfrak{An}_S \rightarrow \mathbf{sets}$ on the category of complex spaces over S such that $P(T)$ (for $T \in \mathfrak{An}_S$) is the set of all isomorphism classes of simple \mathcal{F}_T -coframed sheaves (\mathcal{E}, φ) on X_T/T , where $X_T := X \times_S T$ and \mathcal{F}_T is the pullback of \mathcal{F} to X_T . The main result of this paper is

Theorem 1.1. *If X is cohomologically flat over S in dimension 0, then the functor P is representable by a (not necessarily separated) complex space.*

Thus, informally speaking there is a (relative) moduli space of \mathcal{F} -coframed sheaves on X/S . An inspection of the proof shows that the analogous result holds

1991 *Mathematics Subject Classification.* 32G13, 14D20.

This paper was prepared during a visit of the second author to the University of Bochum which was financed by EAGER - European Algebraic Geometry Research Training Network, contract No. HPRN-CT-2000-00099 (BW 99.0030).

for simple \mathcal{F} -framed sheaves provided that \mathcal{F} is flat over S . The reason that in the case of \mathcal{F} -framed sheaves we need this additional assumption is that only in this case the functor $\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F})$ is known to be representable (see 2.3) whereas $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{E})$ is representable as soon as \mathcal{E} is S -flat.

Our main motivation for studying moduli spaces of \mathcal{F} -(co)framed sheaves is the following. In the case that \mathcal{F} and \mathcal{E} are locally free, a pair (\mathcal{E}, φ) as above is called a \mathcal{F} -(co)framed vector bundle or *holomorphic pair*. Various types of holomorphic pairs over compact complex manifolds (e.g. the coframed ones with $\mathcal{F} = \mathcal{O}_X$ or the framed ones with arbitrary \mathcal{F} , see [OT1], [OT2]) can be identified with solutions of so-called *vortex equations* via a Kobayashi-Hitchin type correspondence. On complex surfaces, these solutions can further be identified with solutions of Seiberg-Witten equations, and moduli spaces \mathcal{M}^{st} of *stable* holomorphic pairs (which are open subsets of the moduli spaces \mathcal{M}^s of simple ones) can be used to effectively calculate Seiberg-Witten invariants in several cases.

It is important to notice that the set \mathcal{M}^s a priori has two analytic structures. One is given by our result and makes it possible to determine \mathcal{M}^s using complex-analytic deformation theory. The other one, which is the one relevant in Seiberg-Witten theory, is given by a gauge theoretical description as in [LL]. But the main result of that paper is in fact that these two structures are indeed the same.

Finally we mention that moduli spaces of stable \mathcal{F} -framed sheaves on *algebraic* manifolds have been constructed in [HL1].

Without a (co)framing, a moduli space of simple bundles was constructed in [KO] and in a more general context in [FS]. In this paper we follow closely the method of proof in the latter paper. The main difficulty is to verify the relative representability of the general criterion 4.3 for the functor P in 1.1. For this we will show in Sect. 2 that the functor of endomorphisms of \mathcal{F} -coframed sheaves is representable. In Sect. 3 we show the openness of the set of points where a coframed sheaf is simple. After these preparations it will be easy to give in Sect. 4 the proof of 1.1.

2. PREPARATIONS

We start with an algebraic lemma.

Lemma 2.1. *Let R be a ring, L, M, N R -modules, and $\alpha : L \rightarrow M$, $\beta : L \rightarrow N$ R -linear maps. If $K := \mathrm{coker}((\alpha, -\beta) : L \rightarrow M \oplus N)$, then*

$$\mathbb{S}(K) \cong \mathbb{S}(M) \otimes_{\mathbb{S}(L)} \mathbb{S}(N)$$

where we consider the symmetric algebras $\mathbb{S}(M)$, $\mathbb{S}(N)$ as $\mathbb{S}(L)$ -algebras via the maps $\mathbb{S}(\alpha)$, $\mathbb{S}(\beta)$.

Proof. There are canonical maps $M \rightarrow K$ and $N \rightarrow K$ given by

$$m \mapsto \text{residue class of } (m, 0), \quad n \mapsto \text{residue class of } (0, n).$$

These maps induce a commutative diagram

$$\begin{array}{ccc} \mathbb{S}(L) & \longrightarrow & \mathbb{S}(M) \\ \downarrow & & \downarrow \\ \mathbb{S}(N) & \longrightarrow & \mathbb{S}(K) \end{array}$$

and so, using the universal property of the tensor product, there is a natural map

$$\mathbb{S}(M) \otimes_{\mathbb{S}(L)} \mathbb{S}(N) \longrightarrow \mathbb{S}(K).$$

Conversely, to construct a natural map in the other direction, note that the first graded piece of $\mathbb{S}(M) \otimes_{\mathbb{S}(L)} \mathbb{S}(N)$ is just $(M \oplus N)/L = K$, so there is an induced map

$$\mathbb{S}(K) \longrightarrow \mathbb{S}(M) \otimes_{\mathbb{S}(L)} \mathbb{S}(N).$$

It is easy to check that these maps are inverse to each other. \square

Now let S be a complex analytic space, and let \mathfrak{An}_S be the category of analytic spaces over S . Recall that for a coherent \mathcal{O}_S -module \mathcal{F} over S the linear fibre space $\mathbb{V}(\mathcal{F})$ represents the functor

$$F : \mathfrak{An}_S \ni T \mapsto F(T) := \text{Hom}(\mathcal{F}_T, \mathcal{O}_T)$$

(see [Fi] or [EGA, II 1.7]). Note that $F(T)$ has the structure of a $\Gamma(T, \mathcal{O}_T)$ -module. Moreover, if \mathcal{G} is another coherent \mathcal{O}_S -module and $T \mapsto G(T)$ is the associated functor as above, then a transformation of functors $F \longrightarrow G$ will be called *linear* if $F(T) \longrightarrow G(T)$ is $\Gamma(T, \mathcal{O}_T)$ -linear for all $T \in \mathfrak{An}_S$. The reader may easily verify that there is a one-to-one correspondence between such linear transformations of functors and morphisms of sheaves $\mathcal{G} \longrightarrow \mathcal{F}$.

Proposition 2.2. *Let $H, F, G : \mathfrak{An}_S \longrightarrow \mathbf{sets}$ be functors that are represented by linear fibre spaces $\mathbb{V}(\mathcal{H}), \mathbb{V}(\mathcal{F}), \mathbb{V}(\mathcal{G})$, respectively. Let $H \longrightarrow G$ and $F \longrightarrow G$ be linear morphisms of functors, and let $K := H \times_G F$ be the fibered product. Then K is represented by $\mathbb{V}(\mathcal{K})$ with*

$$\mathcal{K} := \text{coker}((\alpha, -\beta) : \mathcal{G} \longrightarrow \mathcal{H} \times \mathcal{F}),$$

where $\alpha : \mathcal{G} \longrightarrow \mathcal{H}$ and $\beta : \mathcal{G} \longrightarrow \mathcal{F}$ are the morphisms of sheaves corresponding to $H \longrightarrow G$ and $F \longrightarrow G$.

Proof. The spaces $\mathbb{V}(\mathcal{H}), \mathbb{V}(\mathcal{F}), \mathbb{V}(\mathcal{G})$ are the analytic spectra associated to the symmetric algebras $\mathbb{S}(\mathcal{H}), \mathbb{S}(\mathcal{F}), \mathbb{S}(\mathcal{G})$, respectively, and $H \times_G F$ is represented by $\mathbb{V}(\mathcal{H}) \times_{\mathbb{V}(\mathcal{G})} \mathbb{V}(\mathcal{F})$ which is the analytic spectrum of $\mathbb{S}(\mathcal{H}) \otimes_{\mathbb{S}(\mathcal{G})} \mathbb{S}(\mathcal{F})$. Hence we need to verify that there is a natural isomorphism

$$\mathbb{S}(\mathcal{H}) \otimes_{\mathbb{S}(\mathcal{G})} \mathbb{S}(\mathcal{F}) \longrightarrow \mathbb{S}(\mathcal{K}),$$

but this is a consequence of Lemma 2.1. \square

Let $f : X \longrightarrow S$ be a fixed proper morphism of complex spaces, and let \mathcal{E}, \mathcal{F} be coherent \mathcal{O}_X -modules, where \mathcal{E} is flat over S . Let

$$H := \underline{\text{Hom}}(\mathcal{F}, \mathcal{E}) : \mathfrak{An}_S \longrightarrow \mathbf{sets}$$

be the functor given by

$$H(T) := \text{Hom}_{X_T}(\mathcal{F}_T, \mathcal{E}_T),$$

where $X_T := X \times_S T$ and $\mathcal{E}_T, \mathcal{F}_T$ are the pullbacks of \mathcal{E}, \mathcal{F} on X_T . We recall the following fact.

Theorem 2.3. *The functor H is representable by a linear fibre space over S .*

For a proof see e.g. [Fl2, 3.2] or [Bi].

Now let $\varphi : \mathcal{F} \longrightarrow \mathcal{E}$ and $\varphi' : \mathcal{F} \longrightarrow \mathcal{E}'$ be fixed morphisms of coherent sheaves on X . Let us consider the functor

$$M = \underline{\mathrm{Hom}}((\mathcal{E}, \varphi), (\mathcal{E}', \varphi')) : \mathfrak{An}_S \longrightarrow \mathbf{sets}$$

defined as follows: For $T \in \mathfrak{An}_S$ the elements of $M(T)$ are the pairs

$$(c, \alpha) \in \Gamma(\mathcal{O}_{X_T}) \times \mathrm{Hom}_{X_T}(\mathcal{E}_T, \mathcal{E}'_T)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{E} \\ c \cdot \mathrm{id}_{\mathcal{F}} \downarrow & & \downarrow \alpha \\ \mathcal{F} & \xrightarrow{\varphi'} & \mathcal{E}' \end{array}$$

commutes, i.e. such that $c \cdot \varphi' = \alpha \cdot \varphi$.

Proposition 2.4. *If \mathcal{O}_X , \mathcal{E} and \mathcal{E}' are flat over S , then M is representable by a linear fibre space $\mathbb{V}(\mathcal{M})$ over S .*

Proof. By Theorem 2.3 the functors

$$F := \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E}') , \quad H := \underline{\mathrm{Hom}}(\mathcal{O}_X, \mathcal{O}_X) , \quad G := \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{E}') .$$

are representable by linear fibre spaces over S . There are natural maps

$$H \longrightarrow G , \quad c \mapsto c \cdot \varphi' ,$$

and

$$F \longrightarrow G , \quad \alpha \mapsto \alpha \circ \varphi .$$

By definition we have

$$M = F \times_G H ,$$

so the result follows from Proposition 2.2. \square

An important property of the sheaf \mathcal{M} in Proposition 2.4 is given by

Lemma 2.5. *The following are equivalent.*

- (a) \mathcal{M} is locally free.
- (b) For every complex space $T \in \mathfrak{An}_S$ the canonical map

$$f_*(\mathrm{Hom}((\mathcal{E}, \varphi), (\mathcal{E}', \varphi'))) \otimes_{\mathcal{O}_S} \mathcal{O}_T \longrightarrow f_{T*}(\mathrm{Hom}((\mathcal{E}_T, \varphi_T), (\mathcal{E}'_T, \varphi'_T)))$$

is an isomorphism.

Moreover, if one of these conditions holds then

$$(1) \quad \mathcal{M} \cong [f_*(\mathrm{Hom}((\mathcal{E}, \varphi), (\mathcal{E}', \varphi')))]^\vee .$$

Proof. First note that for every complex space $T \in \mathfrak{An}_S$ we have

$$(2) \quad (\mathcal{M}_T)^\vee \cong f_{T*}(\mathrm{Hom}((\mathcal{E}_T, \varphi_T), (\mathcal{E}'_T, \varphi'_T))) .$$

Applying this to the case $T = S$, (1) follows immediately from the assumption that \mathcal{M} is locally free. Moreover, if (a) is satisfied we have

$$\mathcal{M}_T^\vee \cong \mathcal{M}^\vee \otimes_{\mathcal{O}_S} \mathcal{O}_T \cong f_*(\mathrm{Hom}((\mathcal{E}, \varphi), (\mathcal{E}', \varphi'))) \otimes_{\mathcal{O}_S} \mathcal{O}_T ,$$

where for the last isomorphism we have used (1). Thus (b) follows.

Conversely, assume that (b) holds. Using (1) we infer from the isomorphism in (b) that

$$\mathrm{Hom}(\mathcal{M}_T, \mathcal{O}_T) \cong \mathrm{Hom}(\mathcal{M}, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{O}_T .$$

Applying this to $T = \{s\}$, $s \in S$ a reduced point, it follows that the map

$$\mathcal{H}om(\mathcal{M}, \mathcal{O}_S) \longrightarrow \mathcal{H}om(\mathcal{M}, \mathcal{O}_S/\mathfrak{m}_s)$$

is surjective for every point $s \in S$. Using standard arguments (see e.g. [EGA, 7.5.2]) we conclude that the functor $\mathcal{H}om(\mathcal{M}, -)$ is exact on the category of coherent \mathcal{O}_S -modules whence \mathcal{M} is locally free, as required. \square

3. SIMPLE \mathcal{F} -COFRAMED SHEAVES

As before let $f : X \longrightarrow S$ be a proper morphism of complex spaces, and \mathcal{F} a fixed \mathcal{O}_X -module. We consider \mathcal{F} -coframed sheaves (\mathcal{E}, φ) on X/S , i.e. \mathcal{E} is a S -flat coherent sheaf on X and $\varphi : \mathcal{F} \longrightarrow \mathcal{E}$ is a morphism of \mathcal{O}_X -modules.

Definition 3.1. (\mathcal{E}, φ) is called *simple at $s \in S$* if its fibre $(\mathcal{E}(s), \varphi(s))$ is simple, i.e. if

$$\begin{array}{ccc} \mathcal{F}(s) & \xrightarrow{\varphi(s)} & \mathcal{E}(s) \\ c \cdot \text{id}_{\mathcal{F}(s)} \downarrow & & \downarrow \alpha \\ \mathcal{F}(s) & \xrightarrow{\varphi(s)} & \mathcal{E}(s) \end{array}$$

is a commutative diagram, then $\alpha = c \cdot \text{id}_{\mathcal{E}(s)}$. Moreover, (\mathcal{E}, φ) is said to be *simple over S* if it is simple at every point.

Notice that this definition of simpleness of $(\mathcal{E}(s), \varphi(s))$ coincides with the one given in the introduction. Later on we will need that the points $s \in S$ at which (\mathcal{E}, φ) is simple form an open set in S . For this we need the following considerations.

By Theorem 2.3 and Proposition 2.4 there are coherent \mathcal{O}_S -modules \mathcal{H} and \mathcal{G} such that

$$\underline{\text{End}}(\mathcal{E}, \varphi) := \underline{\text{Hom}}((\mathcal{E}, \varphi), (\mathcal{E}, \varphi)) \quad \text{and} \quad \underline{\text{End}}(\mathcal{O}_X) := \underline{\text{Hom}}(\mathcal{O}_X, \mathcal{O}_X)$$

are represented by $\mathbb{V}(\mathcal{H})$ resp. $\mathbb{V}(\mathcal{G})$. Let

$$\tilde{a} : \mathcal{G} \longrightarrow \mathcal{H} \quad \text{and} \quad \tilde{b} : \mathcal{H} \longrightarrow \mathcal{G}$$

be the \mathcal{O}_S -linear maps associated to the canonical morphisms of functors

$$\begin{array}{ccc} a : \underline{\text{End}}(\mathcal{E}, \varphi) & \longrightarrow & \underline{\text{End}}(\mathcal{O}_X) , \quad \text{resp.} \quad b : \underline{\text{End}}(\mathcal{O}_X) \longrightarrow \underline{\text{End}}(\mathcal{E}, \varphi) , \\ (c, \alpha) & \longmapsto & c \quad \quad \quad c \longmapsto (c, c \cdot \text{id}_{\mathcal{E}}) \end{array}$$

As $a \circ b = \text{id}_{\underline{\text{End}}(\mathcal{O}_X)}$ we have $\tilde{b} \circ \tilde{a} = \text{id}_{\mathcal{G}}$. In other words, \mathcal{G} is a direct summand of \mathcal{H} so that $\mathcal{H} \cong \mathcal{G} \oplus \mathcal{G}'$ for some coherent sheaf \mathcal{G}' on S .

Lemma 3.2. *The following are equivalent.*

- (1) (\mathcal{E}, φ) is simple on S .
- (2) $\mathcal{G}' = 0$.
- (3) The canonical morphism of functors $b : \underline{\text{End}}(\mathcal{O}_X) \longrightarrow \underline{\text{End}}(\mathcal{E}, \varphi)$ is an isomorphism.

Proof. The functor $\underline{\text{End}}(\mathcal{E}(s), \varphi(s))$ resp. $\underline{\text{End}}(\mathcal{O}_{X(s)})$ on the category \mathfrak{An} of all analytic spaces is represented by $\mathcal{H}(s)$ resp. $\mathcal{G}(s)$. Thus $(\mathcal{E}(s), \varphi(s))$ is simple if and only if $\mathcal{G}(s) \cong \mathcal{H}(s)$ which is equivalent to the vanishing of $\mathcal{G}'(s)$. Using Nakayama's lemma, the equivalence of (1) and (2) follows. Finally, the equivalence of (2) and (3) is immediate from the definition of \mathcal{G}' . \square

Corollary 3.3. *If \mathcal{E} is S -flat, then the set of points $s \in S$ at which (\mathcal{E}, φ) is simple, is an open subset of S .*

Proof. The set of points $s \in S$ for which $\mathcal{G}'/\mathfrak{m}_s \cdot \mathcal{G}' = 0$ is just the complement of the support of \mathcal{G}' and hence Zariski-open in S . Using Lemma 3.2 we get the desired result. \square

Recall that a morphism $f : X \rightarrow S$ of complex spaces is said to be *cohomologically flat in dimension 0* if it is flat and if for every $s \in S$ the natural map $f_*(\mathcal{O}_X) \rightarrow f_*(\mathcal{O}_{X(s)})$ is surjective.

Corollary 3.4. *If (\mathcal{E}, φ) is simple and $f : X \rightarrow S$ is cohomologically flat in dimension 0, then $f_*(\mathcal{O}_X)$ is a locally free \mathcal{O}_S -module, and the functor $\underline{\text{End}}(\mathcal{E}, \varphi)$ is represented by $\mathbb{V}(f_*(\mathcal{O}_X)^\vee)$.*

Proof. The fact that $f_*(\mathcal{O}_X)$ is locally free over S , is well known (see, e.g. [FS, 9.7]). Moreover, since (\mathcal{E}, φ) is simple we have $\underline{\text{End}}(\mathcal{E}, \varphi) = \underline{\text{End}}(\mathcal{O}_X)$. As

$$(f_T)_*(\mathcal{O}_{X_T}) \cong f_*(\mathcal{O}_X) \otimes_{\mathcal{O}_S} \mathcal{O}_T$$

we get

$$\text{Hom}_{X_T}(\mathcal{O}_{X_T}, \mathcal{O}_{X_T}) \cong \Gamma(X_T, \mathcal{O}_{X_T}) \cong \text{Hom}_T(f_*(\mathcal{O}_X)^\vee \otimes_{\mathcal{O}_S} \mathcal{O}_T, \mathcal{O}_T),$$

so the space $\mathbb{V}(f_*(\mathcal{O}_X)^\vee)$ represents $\underline{\text{End}}(\mathcal{E}, \varphi)$ as desired. \square

4. PROOF OF THEOREM 1.1

An *isomorphism* of two \mathcal{F} -coframed sheaves (\mathcal{E}, φ) and (\mathcal{E}', φ') is an isomorphism $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$ such that

$$\begin{array}{ccc} & \mathcal{E} & \\ \varphi \nearrow & \downarrow \alpha & \\ \mathcal{F} & & \mathcal{E}' \\ & \searrow \varphi' & \end{array}$$

commutes. We note that (\mathcal{E}, φ) and (\mathcal{E}', φ') are isomorphic if and only if there is a pair

$$(c, \alpha) \in \Gamma(X, \mathcal{O}_X) \times \text{Hom}_X(\mathcal{E}, \mathcal{E}')$$

such that

- (a) c is a unit in $f_*(\mathcal{O}_X)$,
- (b) α is an isomorphism,
- (c) $\alpha \circ \varphi = c \cdot \varphi'$.

Notice that a simple pair (\mathcal{E}, φ) has no automorphism besides $\text{id}_{\mathcal{E}}$. If S is a reduced point then the converse also holds, i.e. (\mathcal{E}, φ) is simple if and only if $\text{id}_{\mathcal{E}}$ is its only automorphism.

Theorem 4.1. *Assume that X is cohomologically flat over S in dimension 0, and let (\mathcal{E}, φ) and (\mathcal{E}', φ') be simple pairs. Then the functor*

$$F : \mathfrak{A}n_S \rightarrow \mathbf{sets}, \quad F(T) := \begin{cases} \{1\} & \text{if } (\mathcal{E}, \varphi) \cong (\mathcal{E}', \varphi'), \\ \emptyset & \text{otherwise,} \end{cases}$$

is representable by a locally closed subspace of S .

Proof. As the sheaf $f_*(\mathcal{O}_X)$ is locally free over S we may assume that it has constant rank, say, r over \mathcal{O}_S . By Proposition 2.4 the functors

$$M := \underline{\mathrm{Hom}}((\mathcal{E}, \varphi), (\mathcal{E}', \varphi')) , \quad M' := \underline{\mathrm{Hom}}((\mathcal{E}', \varphi'), (\mathcal{E}, \varphi))$$

are representable by linear fibre spaces $\mathbb{V}(\mathcal{M})$ resp. $\mathbb{V}(\mathcal{M}')$, where \mathcal{M} and \mathcal{M}' are coherent $f_*(\mathcal{O}_X)$ -modules. If for some space $T \in \mathfrak{An}_S$ the pairs $(\mathcal{E}_T, \varphi_T)$ and $(\mathcal{E}'_T, \varphi'_T)$ are isomorphic, then by Corollary 3.4 \mathcal{M}_T and \mathcal{M}'_T are locally free \mathcal{O}_T -modules of rank r on T . Thus applying [FS, 9.10] as in the proof of [FS, 9.9], we are reduced to the case that \mathcal{M} and \mathcal{M}' are locally free \mathcal{O}_S -modules of rank r . Let us consider the pairings

$$\begin{aligned} M \times M' &\longrightarrow \underline{\mathrm{End}}(\mathcal{E}, \varphi) , \quad ((c, \alpha), (d, \beta)) \mapsto (cd, \beta \circ \alpha), \\ M' \times M &\longrightarrow \underline{\mathrm{End}}(\mathcal{E}', \varphi') , \quad ((d, \beta), (c, \alpha)) \mapsto (cd, \alpha \circ \beta); \end{aligned}$$

these correspond to pairings

$$\begin{aligned} \mathcal{M}^\vee \otimes \mathcal{M}'^\vee &\xrightarrow{\gamma} f_*(\mathcal{O}_X), \\ \mathcal{M}'^\vee \otimes \mathcal{M}^\vee &\xrightarrow{\gamma'} f_*(\mathcal{O}_X). \end{aligned}$$

Using Lemma 2.5 it follows as in the proof of [FS, 9.9] that our functor F is represented by the open subset

$$S' := S \setminus \mathrm{supp}(\mathrm{coker}(\gamma) \oplus \mathrm{coker}(\gamma')).$$

□

Now we consider the groupoid $\mathfrak{P} \rightarrow \mathfrak{An}_S$, where for $T \in \mathfrak{An}_S$ the objects in $\mathfrak{P}(T)$ are the \mathcal{F}_T -coframed sheaves (\mathcal{E}, φ) on X_T/T , where $\varphi : \mathcal{F}_T \rightarrow \mathcal{E}$ is \mathcal{O}_{X_T} -linear. For $(\mathcal{E}, \varphi) \in \mathfrak{P}(T)$ and $(\mathcal{E}', \varphi') \in \mathfrak{P}(T')$, a morphism $(\mathcal{E}, \varphi) \rightarrow (\mathcal{E}', \varphi')$ is a pair (f, α) , where $f : T' \rightarrow T$ is an S -morphism and $\alpha : f^*(\mathcal{E}) \rightarrow \mathcal{E}'$ is an isomorphism of coherent sheaves such that the diagram

$$\begin{array}{ccc} f^*(\mathcal{F}_T) & \xrightarrow{f^*(\varphi)} & f^*(\mathcal{E}) \\ \parallel & & \downarrow \alpha \\ \mathcal{F}_{T'} & \xrightarrow{\varphi'} & \mathcal{E}' \end{array}$$

commutes.

Proposition 4.2.

- (a) Every object $(\mathcal{E}_0, \varphi_0)$ in $\mathfrak{P}(s)$, $s \in S$, admits a semiuniversal deformation.
- (b) Versality is open in \mathfrak{P} .

Proof. Let $\mathfrak{Q} \rightarrow \mathfrak{An}_S$ be the groupoid where the objects over a space $T \in \mathfrak{An}_S$ are the coherent \mathcal{O}_{X_T} -modules that are T -flat. As usual, given $\mathcal{E} \in \mathfrak{Q}(T)$ and $\mathcal{E}' \in \mathfrak{Q}(T')$, a morphism $\mathcal{E} \rightarrow \mathcal{E}'$ in \mathfrak{Q} consists of a pair (f, α) , where $f : T' \rightarrow T$ is an S -morphism and $\alpha : f^*(\mathcal{E}) \rightarrow \mathcal{E}'$ is an isomorphism of coherent sheaves. Assigning to a pair (\mathcal{E}, φ) the sheaf \mathcal{E} gives a functor $\mathfrak{P} \rightarrow \mathfrak{Q}$. It is well known that there are semiuniversal deformations in \mathfrak{Q} (see [ST] or [BK]) and that versality is open in \mathfrak{Q} (see e.g., [Fl1]).

The fibre of $\mathfrak{P} \rightarrow \mathfrak{Q}$ over a given object $\mathcal{E} \in \mathfrak{Q}(T)$ is the groupoid $\mathfrak{P}_\mathcal{E} \rightarrow \mathfrak{An}_T$ as explained in [Bi, Sect. 10]. More concretely, given a space $Z \in \mathfrak{An}_T$, an object in $\mathfrak{P}_\mathcal{E}$ over Z is a morphism

$$\varphi : \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_Z \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{O}_Z.$$

As the functor underlying $\mathfrak{P}_{\mathcal{E}}$ is representable by Theorem 2.3 we get that the objects in $\mathfrak{P}_{\mathcal{E}}(t)$, $t \in T$, admit semiuniversal deformations and that versality is open in $\mathfrak{P}_{\mathcal{E}}$. Applying [Bi, 10.12] gives the desired conclusion. \square

Before proving the main theorem we remind the reader of the following criterion for the representability of a functor which we present for our purposes in the form as given in [FS, 7.5]; see also [Bi, 3.1] or [KO, §2].

Theorem 4.3. *A functor $F : \mathfrak{An}_S \rightarrow \mathbf{sets}$ is representable by a complex space over S (resp. a separated complex space over S) if and only if the following conditions are satisfied.*

(1) (Existence of semiuniversal deformations) *Every $a_0 \in F(s)$, $s \in S$, admits a semiuniversal deformation.*

(2) (Sheaf axiom) *F is of local nature, i.e. for every complex space $T \in \mathfrak{An}_S$ the presheaf $T \supseteq U \mapsto F(U)$ on T is a sheaf.*

(3) (Relative representability) *For every $T \in \mathfrak{An}_S$ and $a, b \in F(T)$ the set-valued functor $\text{Equ}(a, b)$ with*

$$\text{Equ}(a, b)(Z) := \begin{cases} \{1\} & \text{if } a_Z = b_Z, \\ \emptyset & \text{otherwise,} \end{cases}$$

is representable by a locally closed (resp. closed) subspace of T .

(4) (Openness of versality) *For every $T \in \mathfrak{An}_S$ and $a \in F(T)$ the set of points $t \in T$ at which a is formally versal is open in T .*

Proof of Theorem 1.1. We will verify that the conditions (1)–(4) in Theorem 4.3 are satisfied. (1) and (4) hold by Proposition 4.2. Moreover, (3) is just Theorem 4.1. Finally, (2) holds as simple pairs have no non-trivial automorphism. \square

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FAKULTÄT FÜR MATHEMATIK DER RUHR-UNIVERSITÄT, UNIVERSITÄTSSTR. 150, GEB. NA 2/72,
44780 BOCHUM, GERMANY

E-mail address: `Hubert.Flenner@ruhr-uni-bochum.de`

MATHEMATICAL INSTITUTE, LEIDEN UNIVERSITY, PO BOX 9512, NL-2300 RA LEIDEN, THE
NETHERLANDS

E-mail address: `lubke@math.leidenuniv.nl`